

GENUS 2 PARAMODULAR EISENSTEIN CONGRUENCES

DAN FRETWELL

Abstract

We investigate certain Eisenstein congruences, as predicted by Harder, for level p paramodular forms of genus 2. We use algebraic modular forms to generate new evidence for the conjecture. In doing this we see explicit computational algorithms that generate Hecke eigenvalues for such forms.

1. INTRODUCTION

Congruences between modular forms have been found and studied for many years. Perhaps the first interesting example is found in the work of Ramanujan. He studied in great detail the Fourier coefficients $\tau(n)$ of the discriminant function $\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}$ (where $q = e^{2\pi iz}$). The significance of Δ is that it is the unique normalized cusp form of weight 12.

Amongst Ramanujan's mysterious observations was a pretty congruence:

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691}.$$

Here $\sigma_{11}(n) = \sum_{d|n} d^{11}$ is a power divisor sum. Naturally one wishes to explain the appearance of the modulus 691. The true incarnation of this is via the fact that the prime 691 divides the numerator of the “rational part” of $\zeta(12)$, i.e. $\frac{\zeta(12)}{\pi^{12}} \in \mathbb{Q}$ (a quantity that appears in the Fourier coefficients of the Eisenstein series E_{12}).

Since the work of Ramanujan there have been many generalizations of his congruences. Indeed by looking for big enough primes dividing numerators of normalized zeta values one can provide similar congruences at level 1 between cusp forms and Eisenstein series for other weights. In fact one can even give “local origin” congruences between level p cusp forms and level 1 Eisenstein series by extending the divisibility criterion to include single Euler factors of $\zeta(s)$ rather than the global values of $\zeta(s)$ (see [7] for results and examples).

There are also Eisenstein congruences predicted for Hecke eigenvalues of genus 2 Siegel cusp forms. One particular type was conjectured to exist by Harder [14]. There is only a small amount of evidence for this conjecture, the literature only contains examples at levels 1 and 2 (using methods specific to these levels). The conjecture is also far from being proved. Only one specific level 1 example of the congruence has been proved (p.386 of [5]).

In this paper we will see new evidence for a level p version of Harder's conjecture for various small primes (including $p = 2$ but not exclusively). The Siegel forms will be of paramodular type and the elliptic forms will be of $\Gamma_0(p)$ type. In doing this we will make use of Jacquet-Langlands style conjectures due to Ibukiyama.

Dan Fretwell, Heilbronn Institute for Mathematical Research, School of Mathematics, University of Bristol, U.K. Email: daniel.fretwell@bristol.ac.uk.

2. HARDER'S CONJECTURE

Given $k \geq 0$ and $N \geq 1$ let $S_k(\Gamma_0(N))$ denote the space of elliptic cusp forms for $\Gamma_0(N)$. Also for $j \geq 0$ let $S_{j,k}(K(N))$ denote the space of genus 2, vector-valued Siegel cusp forms for the paramodular group of level N , taking values in the representation space $\text{Sym}^j(\mathbb{C}^2) \otimes \det^k$ of $\text{GL}_2(\mathbb{C})$.

Given $f \in S_k(\Gamma_0(N))$ we let $\Lambda_{\text{alg}}(f, j+k) = \frac{\Lambda(f, j+k)}{\Omega}$, where $\Lambda(f, s)$ is the completed L-function attached to f and Ω is a Deligne period attached to f . The choice of Ω is unique up to scaling by \mathbb{Q}_f^\times but Harder shows how to construct a more canonical choice of Ω that is determined up to scaling by $\mathcal{O}_{\mathbb{Q}_f}^\times$ [15].

In this paper we consider the following paramodular version of Harder's conjecture (when $N = 1$ this is the original conjecture found in [14]).

Conjecture 2.1. *Let $j > 0$, $k \geq 3$ and let $f \in S_{j+2k-2}^{\text{new}}(\Gamma_0(N))$ be a normalized Hecke eigenform with eigenvalues a_n . Suppose that $\text{ord}_\lambda(\Lambda_{\text{alg}}(f, j+k)) > 0$ for some prime λ of \mathbb{Q}_f lying above a rational prime $l > j+2k-2$ (with $l \nmid N$).*

Then there exists a Hecke eigenform $F \in S_{j,k}^{\text{new}}(K(N))$ with eigenvalues $b_n \in \mathbb{Q}_F$ such that

$$b_q \equiv q^{k-2} + a_q + q^{j+k-1} \pmod{\Lambda}$$

for all primes $q \nmid N$ (where Λ is some prime lying above λ in the compositum $\mathbb{Q}_f\mathbb{Q}_F$).

It should be noted that Harder's conjecture has still not been proved for level 1 forms. However the specific example with $j = 4, k = 10$ and $l = 41$ mentioned in Harder's paper has recently been proved in a paper by Chenevier and Lannes [5]. The proof uses the Niemeier classification of 24-dimensional lattices and is specific to this particular case.

Following the release of the level 1 conjecture, Faber and Van der Geer were able to do computations when $\dim(S_{j,k}(\text{Sp}_4(\mathbb{Z}))) = 1$. They have now exhausted such spaces and in each case have verified the congruence for a significant number of Hecke eigenvalues. Extra evidence for the case $j = 2$ is given by Ghitza, Ryan and Sulon [11].

For the level p conjecture a substantial amount of evidence has been provided by Bergström et al for level 2 forms [1]. Their methods are specific to this level. A small amount of evidence is known beyond level 2. In particular a congruence has been found with $(j, k, p, l) = (0, 3, 61, 43)$ by Anton Mellit (p.99 of [15]).

In this paper we use the theory of algebraic modular forms to provide evidence for the conjecture at levels $p = 2, 3, 5, 7$. The methods discussed can be extended to work for other levels.

3. ALGEBRAIC MODULAR FORMS

In general, it is quite tough to compute Hecke eigensystems for paramodular forms. Fortunately for a restricted set of levels there is a (conjectural) Jacquet-Langlands style correspondence for GSp_4 due to Ihara and Ibukiyama [18].

Explicitly it is expected that there is a Hecke equivariant isomorphism between the spaces $S_{j,k}^{\text{new}}(K(p))$ and certain spaces of algebraic modular forms. Bearing this in mind we give the reader a brief overview of the general theory of such forms.

3.1. The spaces $\mathcal{A}(G, K_f, V)$ of algebraic forms. Let G/\mathbb{Q} be a connected reductive group with the added condition that the Lie group $G(\mathbb{R})$ is connected and compact modulo center. Fix an open compact subgroup $K_f \subset G(\mathbb{A}_f)$. Also let V be (the space of) a finite dimensional algebraic representation of G , defined over a number field F .

Definition 3.1. The F -vector space of *algebraic modular forms* of level K_f , weight V for G is:

$$\mathcal{A}(G, K_f, V) \cong \{f : G(\mathbb{A}_f) \rightarrow V \mid f(\gamma g k) = \gamma f(g), \forall (\gamma, g, k) \in G(\mathbb{Q}) \times G(\mathbb{A}) \times K_f\}.$$

Fix a set of representatives $T = \{z_1, z_2, \dots, z_h\} \in G(\mathbb{A}_f)$ for $G(\mathbb{Q}) \backslash G(\mathbb{A}_f) / K_f$. There is a natural embedding:

$$\begin{aligned} \phi : \mathcal{A}(G, K_f, V) &\longrightarrow V^h \\ f &\longmapsto (f(z_1), \dots, f(z_h)). \end{aligned}$$

Theorem 3.2. *The map ϕ induces an isomorphism:*

$$\mathcal{A}(G, K_f, V) \cong \bigoplus_{m=1}^h V^{\Gamma_m},$$

where $\Gamma_m = G(\mathbb{Q}) \cap z_m K_f z_m^{-1}$ for each m .

Corollary 3.3. *The spaces $\mathcal{A}(G, K_f, V)$ are finite dimensional.*

A pleasing feature of the theory is that the groups Γ_m are often finite. Gross gives many equivalent conditions for when this happens [13]. One such condition is the following.

Proposition 3.4. *The groups Γ_m are finite if and only if $G(\mathbb{Z})$ is finite.*

3.2. Hecke Operators. Let $u \in G(\mathbb{A}_f)$ and fix a decomposition $K_f u K_f = \coprod_{i=1}^r u_i K_f$. It is well known that finitely many representatives occur. Then T_u acts on $f \in \mathcal{A}(G, K_f, V)$ via

$$T_u(f)(g) := \sum_{i=1}^r f(g u_i), \quad \forall g \in G(\mathbb{A}_f).$$

It is easy to see that this is independent of the choice of representatives u_i since they are determined up to right multiplication by K_f .

We wish to find the Hecke representatives u_i explicitly and efficiently. To this end a useful observation can be made when the class number is one.

Proposition 3.5. *If $h = 1$ then we may choose Hecke representatives that lie in $G(\mathbb{Q})$.*

Finally we note that for G satisfying Proposition 3.4 there is a natural inner product on the space $\mathcal{A}(G, K, V)$. This is given in Gross' paper [13] but we shall give the rough details here.

Lemma 3.6. *Let G satisfy the property of Proposition 3.4 and V be a finite dimensional algebraic representation of G , defined over \mathbb{Q} . Then there exists a character $\mu : G \rightarrow \mathbb{G}_m$ and a positive definite symmetric bilinear form $\langle, \rangle : V \times V \rightarrow \mathbb{Q}$ such that:*

$$\langle \gamma u, \gamma v \rangle = \mu(\gamma) \langle u, v \rangle$$

for all $\gamma \in G(\mathbb{Q})$.

Taking adelic points we have a character $\mu' : G(\mathbb{A}) \rightarrow \mathbb{A}^\times$. Let $\mu_{\mathbb{A}} = f \circ \mu'$, where $f : \mathbb{A}^\times \rightarrow \mathbb{Q}^\times$ is the natural projection map coming from the decomposition $\mathbb{A}^\times = \mathbb{Q}^\times \mathbb{R}^+ \hat{\mathbb{Z}}^\times$.

Proposition 3.7. *Let G satisfy the property of Proposition 3.4. Then $\mathcal{A}(G, K, V)$ has a natural inner product given by:*

$$\langle f, g \rangle = \sum_{m=1}^h \frac{1}{|\Gamma_m| \mu_{\mathbb{A}}(z_m)} \langle f(z_m), g(z_m) \rangle.$$

3.3. Trace of Hecke operators. The underlying representation V of G is typically big in dimension and so the action of Hecke operators is, although explicit, quite tough to compute. Fortunately, there is a simple trace formula for Hecke operators on spaces of algebraic modular forms. The details of the formula can be found in [8] but we give brief details here.

Note that $G(\mathbb{A}_f)$ acts on the set Z on the left by setting $w \cdot z_i = z_j$ if and only if $G(\mathbb{Q})(wz_i)K_f = G(\mathbb{Q})z_jK_f$. For each $m = 1, 2, \dots, h$ we consider the set $S_m = \{i \mid u_i \cdot z_m = z_m\}$. Next for each $i \in S_m$ choose elements $k_{m,i} \in K_f$ and $\gamma_{m,i} \in G(\mathbb{Q})$ such that $\gamma_{m,i}^{-1} u_i z_m k_{m,i} = z_m$.

Let χ_V denote the character of the representation of $G(\mathbb{Q})$ on V . Then the trace formula is as follows.

Theorem 3.8. (*Dummigan*)

$$\text{tr}(T_u) = \sum_{m=1}^h \frac{1}{|\Gamma_m|} \sum_{\gamma \in \Gamma_m, i \in S_m} \chi_V(\gamma_{m,i} \gamma).$$

More generally:

$$\text{tr}(T_u^d) = \sum_{m=1}^h \frac{1}{|\Gamma_m|} \sum_{\gamma \in \Gamma_m, (i_n) \in S_m^d} \chi_V \left(\left(\prod_{n=1}^d \gamma_{m,i_n} \right) \gamma \right).$$

Letting $u = \text{id}$ we recover the following.

Corollary 3.9.

$$\dim(\mathcal{A}(G, K_f, V)) = \sum_{m=1}^h \frac{1}{|\Gamma_m|} \sum_{\gamma \in \Gamma_m} \chi_V(\gamma).$$

When $h = 1$ the situation becomes much simpler. In this case we may choose $z_1 = \text{id}$ and $\gamma_{1,i} = u_i \in G(\mathbb{Q})$ for each i (this is possible by Corollary 3.5).

Corollary 3.10. *If $h = 1$ then we have*

$$\text{tr}(T_u) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma, 1 \leq i \leq r} \chi_V(u_i \gamma),$$

where $\Gamma = G(\mathbb{Q}) \cap K_f$.

The trace formula was introduced to test a $U(2, 2)$ analogue of Harder's conjecture. In this paper we will use it to test the level p paramodular version of Harder's conjecture given by Conjecture 2.1.

4. EICHLER AND IBUKIYAMA CORRESPONDENCES

4.1. Eichler's correspondence. From now on D will denote a quaternion algebra over \mathbb{Q} ramified at $\{p, \infty\}$ (for a fixed prime p) and \mathcal{O} will be a fixed maximal order. Since D is definite, we have that $D_\infty^\times = D^\times \otimes \mathbb{R} \cong \mathbb{H}^\times$ is compact modulo center (and is also connected). Thus we may consider algebraic modular forms for the group $G = D^\times$.

Also note that in this case each Γ_m will be finite since $D^\times(\mathbb{Z}) = \mathcal{O}^\times$ is finite.

Let $D_q := D \otimes \mathbb{Q}_q$ be the local component at prime q (no restriction on q) and let $D_{\mathbb{A}_f}$ be the restricted direct product of the D_q 's with respect to the local maximal orders $\mathcal{O}_q := \mathcal{O} \otimes \mathbb{Z}_q$.

Note that if $q \neq p$ then $D_q^\times \cong (M_2(\mathbb{Q}_q))^\times = \mathrm{GL}_2(\mathbb{Q}_q)$. Thus locally away from the ramified prime, D^\times behaves like GL_2 .

In fact more is true. It is the case that the reductive groups D^\times and GL_2 are inner forms of each other. So by the principle of Langlands functoriality we expect a transfer of automorphic forms between D^\times and GL_2 . Eichler gives an explicit description of this transfer.

Let $V_n = \mathrm{Sym}^n(\mathbb{C}^2)$ (for $n \geq 0$). Then V_n gives a well defined representation of $\mathrm{SU}(2)/\{\pm I\}$ if and only if n is even. Thus we get a well defined action on V_n by D^\times via:

$$D^\times \hookrightarrow \mathbb{H}^\times \longrightarrow \mathbb{H}^\times / \mathbb{R}^\times \cong \mathrm{SU}(2) / \{\pm I\}.$$

Take $U = \prod_q \mathcal{O}_q^\times$. This is an open compact subgroup of $D_{\mathbb{A}_f}^\times$.

Theorem 4.1. (Eichler) *Let $k > 2$. Then there is a Hecke equivariant isomorphism:*

$$S_k^{\mathrm{new}}(\Gamma_0(p)) \cong \mathcal{A}(D^\times, U, V_{k-2}).$$

For $k = 2$ the above holds if on the right we quotient out by the space of constant functions.

It remains to describe how the Hecke operators transfer over the isomorphism. Fix a prime $q \neq p$. Choose $u \in D_{\mathbb{A}_f}^\times$ such that $\psi(u_q) = \mathrm{diag}(1, p)$ and is the identity at all other places. The corresponding Hecke operator $T_{u,q}$ corresponds to the classical T_q operator under Eichler's correspondence.

4.2. Ibukiyama's correspondence. Ibukiyama's correspondence is a (conjectural) generalisation of Eichler's correspondence to Siegel modular forms. The details can be found in [18] but we explain the main ideas.

Given the setup in the previous subsection, consider the unitary similitude group:

$$\mathrm{GU}_n(D) = \{g \in M_n(D) \mid g\bar{g}^T = \mu(g)I, \mu(g) \in \mathbb{Q}^\times\}.$$

Here \bar{g} means componentwise application of the standard involution of D . This group is the similitude group of the standard Hermitian form on D^n .

Theorem 4.2. *For any field K there exists a similitude-preserving isomorphism $GU_2(M_2(K)) \cong GSp_4(K)$.*

Proof. Conjugation by the matrix $M = \text{diag}(1, A, 1) \in M_4(K)$, where $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ gives such an isomorphism. \square

One consequence of this is that the group $GU_2(D)$ behaves like GSp_4 locally away from the ramified prime. It is indeed true that these groups are also inner forms of each other.

A simple argument also shows that $GU_2(\mathbb{H})/Z(GU_2(\mathbb{H})) \cong USp(4)/\{\pm I\}$. Thus $GU_2(D_\infty)$ is compact modulo center and connected. Thus we may consider algebraic modular forms for this group. Once again we are guaranteed that the Γ_m groups are finite by the following.

Lemma 4.3. *$GU_2(\mathcal{O}) = \{\gamma \in GU_2(D) \cap M_2(\mathcal{O}) \mid \mu(\gamma) \in \mathbb{Z}^\times\}$ is finite.*

Proof. Solving the equations gives:

$$GU_2(\mathcal{O}) = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \mid \alpha, \beta \in \mathcal{O}^\times \right\}.$$

\square

One consequence of Theorem 4.2 is that $GU_2(D_q) \cong GSp_4(\mathbb{Q}_q)$ for all $q \neq p$.

Proposition 4.4. *For any $q \neq p$ there exists a similitude-preserving isomorphism $\psi : GU_2(D_q) \rightarrow GSp_4(\mathbb{Q}_q)$ that preserves integrality:*

$$\psi(GU_2(D_q) \cap M_2(\mathcal{O}_q)) = GSp_4(\mathbb{Q}_q) \cap M_4(\mathbb{Z}_q).$$

Proof. Choose an isomorphism of quaternion algebras $D_q \cong M_2(\mathbb{Q}_q)$ that preserves the norm, trace and integrality. This induces an isomorphism with the required properties:

$$GU_2(D_q) \cong GU_2(M_2(\mathbb{Q}_q)) \cong GSp_4(\mathbb{Q}_q).$$

\square

Let $V_{j,k-3}$ be the irreducible representation of $USp(4)$ with Young diagram parameters $(j+k-3, k-3)$. This gives a well defined representation of $USp(4)/\{\pm I\}$ if and only if j is even. Thus $GU_2(D)$ acts on this via:

$$GU_2(D) \hookrightarrow GU_2(\mathbb{H}) \twoheadrightarrow GU_2(\mathbb{H})/Z(GU_2(\mathbb{H})) \cong USp(4)/\{\pm I\}.$$

The groups $GU(D)$ and GSp_4 are inner forms. Thus (as with Eichler) one expects a transfer of automorphic forms. The following is found in Ibukiyama's paper [18].

Conjecture 4.5. *(Ibukiyama) Let $j \geq 0$ be an even integer and $k \geq 3$. Suppose $(j, k) \neq (0, 3)$. Then there is a Hecke equivariant isomorphism:*

$$S_{j,k}^{new}(\Gamma_0(p)) \longrightarrow \mathcal{A}^{new}(GU_2(D), U_1, V_{j,k-3})$$

$$S_{j,k}^{new}(K(p)) \longrightarrow \mathcal{A}^{new}(GU_2(D), U_2, V_{j,k-3}),$$

where $U_1, U_2, V_{j,k-3}$ are to be defined.

If $(j, k) = (0, 3)$ then we also get an isomorphism after taking the quotient by the constant functions on the right.

Since our eventual goal is to study Harder's conjecture for paramodular forms we will neglect the first of these isomorphisms. However, it will turn out that the open compact subgroup U_1 will prove useful in later calculations.

4.2.1. *The levels U_1 and U_2 .* In Eichler's correspondence, the "level 1" open compact subgroup $U = \prod_q \mathcal{O}_q^\times \subset D_{\mathbb{A}_f}^\times$ can be viewed as $\text{Stab}_{D_{\mathbb{A}_f}^\times}(\mathcal{O})$ under an action defined by right multiplication. Similarly, one can produce open compact subgroups $\text{Stab}_{\text{GU}_2(D_{\mathbb{A}_f})}(L) \subseteq \text{GU}_2(D_{\mathbb{A}_f})$, where L is a left \mathcal{O} -lattice of rank 2 in D^2 (a free left \mathcal{O} -module of rank 2).

A left \mathcal{O} -lattice $L \subseteq D^2$ gives rise to a left \mathcal{O}_q -lattice $L_q = L \otimes \mathbb{Z}_q \subseteq D_q^2$ for each prime q . A result of Shimura tells us the possibilities for L_q (see [26]).

Theorem 4.6. *Let D be a quaternion algebra over \mathbb{Q} . If D is split at q then L_q is right $\text{GU}_2(D_q)$ equivalent to \mathcal{O}_q^2 . If D is ramified at q then there are exactly two possibilities for L_q , up to right $\text{GU}_2(D_q)$ equivalence (one being \mathcal{O}_q^2).*

When D is ramified at $\{p, \infty\}$ it is clear from this result that there are only two possibilities for L , up to local equivalence.

Definition 4.7. Let D be ramified at p, ∞ for some prime p :

- If L_p is locally equivalent to \mathcal{O}_p^2 for all q then we say that L lies in the principal genus.
- If L_p is locally inequivalent to \mathcal{O}_p^2 then we say that L lies in the non-principal genus. \square

Given L , results of Ibukiyama [21] allow us to write $L = \mathcal{O}^2 g$ for some $g \in \text{GL}_2(D)$ and determine the genus of L based on g .

Theorem 4.8. *• L lies in the principal genus if and only if $g\bar{g}^T = mx$ for some positive $m \in \mathbb{Q}$ and some $x \in \text{GL}_n(\mathcal{O})$ such that $x = \bar{x}^T$ and such that x is positive definite, i.e. $yx\bar{y}^T > 0$ for all $y \in D^n$ with $y \neq 0$.*

• L lies in the non-principal genus if and only if $g\bar{g}^T = m \begin{pmatrix} ps & r \\ \bar{r} & pt \end{pmatrix}$ where $m \in \mathbb{Q}$ is positive, $s, t \in \mathbb{N}, r \in \mathcal{O}$ lies in the two sided ideal of \mathcal{O} above p and is such that $p^2st - N(r) = p$ (so that the matrix on the right has determinant p).

The lattice \mathcal{O}^2 is clearly in the principal genus and corresponds to the choice $g = I$. Alternatively fix a choice of g such that $\mathcal{O}^2 g$ is in the non-principal genus. Let U_1, U_2 respectively denote the corresponding open compact subgroups of $\text{GU}_2(D_{\mathbb{A}_f})$ (as described above).

4.2.2. *Hecke operators.* The transfer of Hecke operators in Ibukiyama's correspondence is similar to the Eichler correspondence but has subtle differences. Fix a prime $q \neq p$ and let $M_q = \text{diag}(1, 1, q, q) \in \text{GSp}_4(\mathbb{Q}_q)$. Fixing an isomorphism as in Proposition 4.4 we may choose $v_q \in \text{GU}_2(D_q)$ such that $v_q \mapsto M_q$. Since $q \neq p$ we know that $\mathcal{O}_q^2 g_q = \mathcal{O}_q^2 h_q$ for some $h_q \in \text{GU}_2(D_q)$ (where g_q is the image

of g under the natural embedding $\mathrm{GU}_2(D) \rightarrow \mathrm{GU}_2(D_q)$. $u_q \in \mathrm{GU}_2(D_q)$ given by $u_q = h_q v_q h_q^{-1}$.

Let $u \in \mathrm{GU}_2(D_{\mathbb{A}_f})$ have $u_q = h_q v_q h_q^{-1}$ as the component at q and have identity component elsewhere.

Definition 4.9. For the above choice of u , the corresponding Hecke operator on $\mathcal{A}^{\mathrm{new}}(\mathrm{GU}_2(D), U_2, V_{j,k-3})$ will be called $T_{u,q}$. \square

Under Ibukiyama's correspondence it is predicted that $T_{u,q}$ corresponds to the classical T_q operator acting on $S_{j,k}^{\mathrm{new}}(K(p))$.

4.2.3. The new subspace. Our final task in defining Ibukiyama's correspondence is to explain what is meant by the new subspace $\mathcal{A}^{\mathrm{new}}(\mathrm{GU}_2(D), U_2, V_{j,k-3})$. We will not go into too much detail but will refer the reader to Ibukiyama's papers [20], [22].

Let $G = D^\times \times \mathrm{GU}_2(D)$. Then we have an open compact subgroup $U' = U \times U_2$ and finite dimensional representations $W_{j,k-3} := V_j \otimes V_{j,k-3}$ of $G(\mathbb{A}_f)$.

We start with the decomposition:

$$\mathcal{A}(G, U', W_{j,k-3}) \cong \mathcal{A}(D^\times, U, V_j) \otimes \mathcal{A}(\mathrm{GU}_2(D), U_2, V_{j,k-3}).$$

Ibukiyama takes $F \in \mathcal{A}(G, U', W_{j,k-3})$. If F is an eigenform then $F = F_1 \otimes F_2$ for eigenforms F_1, F_2 . He then associates an explicit theta series θ_F to F . This is an elliptic modular form for $SL_2(\mathbb{Z})$ of weight $j + 2k - 2$ (if $j + 2k - 6 \neq 0$ then it is a cusp form). It is known that θ_F is an eigenform for all Hecke operators if and only if $\theta_F \neq 0$.

Definition 4.10. The subspace of old forms $A_{j,k-3}^{\mathrm{old}}(D) \subseteq A_{j,k-3}(D)$ is generated by the eigenforms F_2 such that there exists an eigenform F_1 satisfying $\theta_{F_1 \otimes F_2} \neq 0$.

The subspace of new forms $A_{j,k-3}^{\mathrm{new}}(D)$ is the orthogonal complement of the old space with respect to the inner product in Proposition 3.7. \square

It should be noted that by Eichler's correspondence F_1 can be viewed as an elliptic modular form for $\Gamma_0(p)$ of weight $j + 2$. Further it will be a new cusp form precisely when $j > 0$. Thus computationally it is not difficult to find the new and old subspaces.

5. FINDING EVIDENCE FOR HARDER'S CONJECTURE

Now that we have linked spaces of Siegel modular forms $S_{j,k}^{\mathrm{new}}(K(p))$ with spaces of algebraic modular forms $A_{j,k-3}^{\mathrm{new}}(D) = \mathcal{A}^{\mathrm{new}}(\mathrm{GU}_2(D), U_2, V_{j,k-3})$, we can begin to generate evidence for Harder's conjecture.

5.1. Brief plan of the strategy. In this paper we will deal with cases where $h = 1$ and $\dim(A_{j,k-3}^{\mathrm{new}}(D)) = 1$.

Strategy

- (1) Find all primes p such that $h = 1$.
- (2) For each such p calculate $\Gamma^{(2)} = \mathrm{GU}_2(D) \cap U_2$.
- (3) Using Corollary 3.9 find all j, k such that $\dim(A_{j,k}^{\mathrm{new}}(D)) = 1$.

- (4) For each pair (j, k) look in the space of elliptic forms $S_{j+2k-2}^{\text{new}}(\Gamma_0(p))$ for normalized eigenforms f which have a “large prime” dividing $\Lambda_{\text{alg}}(f, j+k) \in \mathbb{Q}_f$.
- (5) Find the Hecke representatives for the $T_{u,q}$ operator at a chosen prime q .
- (6) Use the trace formula to find $\text{tr}(T_{u,q})$ for T_q acting on $A_{j,k-3}(D)$.
- (7) Subtract off the trace contribution of $T_{u,q}$ acting on $A_{j,k-3}^{\text{old}}(D)$ in order to get the trace of the action on $A_{j,k-3}^{\text{new}}(D)$. Since $\dim(A_{j,k-3}^{\text{new}}(D)) = 1$ this trace should be exactly the Hecke eigenvalue of a new paramodular eigenform by Ibukiyama’s conjecture.
- (8) Check that Harder’s congruence holds.

The above strategy can be modified to work for the case $\dim(A_{j,k-3}^{\text{new}}(D)) = d > 1$ but one must compute $\text{tr}(T_{u,q}^t)$ for $1 \leq t \leq d$.

5.2. Finding $\Gamma^{(2)}$. For $\theta \in \mathbb{Q}^\times$ consider the subset:

$$\text{GU}_n(D)_\theta = \{\gamma \in \text{GU}_2(D) \mid \mu(\gamma) = \theta\},$$

In particular let $\text{SU}_2(D) := \text{GU}_2(D)_1$.

Theorem 5.1. *The group $\Gamma^{(2)}$ consists of the following set of matrices:*

$$\Gamma^{(2)} = \text{SU}_2(D) \cap g^{-1} \text{GL}_2(\mathcal{O})g$$

Proof. We know that:

$$\begin{aligned} \Gamma^{(2)} &= \text{Stab}_{\text{GU}_2(D)}(\mathcal{O}^2g) = \text{GU}_2(D) \cap \text{Stab}_{\text{GL}_2(D)}(\mathcal{O}^2g) \\ &= \text{GU}_2(D) \cap g^{-1}Sg = \text{GU}_2(D) \cap g^{-1}\text{GL}_2(\mathcal{O})g. \end{aligned}$$

A simple calculation shows that any such matrix has similitude 1. \square

Recall also the open compact subgroup $U_1 = \text{Stab}_{\text{GU}_2(\mathbb{A}_f)}(\mathcal{O}^2) \subset \text{GU}_2(D_{\mathbb{A}_f})$. This is the stabilizer of a left \mathcal{O} -lattice lying in the principal genus.

In this case the analogue of the group $\Gamma^{(2)}$ is the group $\Gamma^{(1)} = \text{GU}_2(D) \cap U_1$. We can employ identical arguments to the above to show the following:

Lemma 5.2.

$$\Gamma^{(1)} = \text{SU}_2(D) \cap \text{GL}_2(\mathcal{O}) = \text{GU}_2(\mathcal{O}).$$

We already have an explicit description of $\Gamma^{(1)}$ (see Lemma 4.3). Computationally it is not straight forward to find the elements of $\Gamma^{(2)}$ due to the non-integrality of the entries of such matrices.

For $\theta \in \mathbb{Q}^\times$ consider the sets

$$Y_\theta = \text{GU}_2(D)_\theta \cap g^{-1}M_2(\mathcal{O})^\times g$$

and

$$W_\theta = \{\nu \in M_2(\mathcal{O})^\times \mid \nu A \bar{\nu}^T = \theta A\},$$

where $M_2(\mathcal{O})^\times = \text{GL}_2(D) \cap M_2(\mathcal{O})$ and $A = g\bar{g}^T$.

Then in particular $Y_1 = \Gamma^{(2)}$. Later the sets Y_q for prime $q \neq p$ will appear when finding Hecke representatives.

Proposition 5.3. *For each $\theta \in \mathbb{Q}^\times$ conjugation by g gives a bijection:*

$$\Phi_\theta : Y_\theta \longrightarrow W_\theta.$$

To calculate the sets W_θ we diagonalize A . Choose a matrix $P \in \text{GL}_2(D)$ such that $PA\overline{P}^T = B$ where $B \in M_2(D)$ is a diagonal matrix.

Proposition 5.4. *For each $\theta \in \mathbb{Q}^\times$ conjugation by P gives a bijection*

$$W_\theta \longrightarrow Z_\theta := \{\eta \in P M_2(\mathcal{O})^\times P^{-1} \mid \eta B \overline{\eta}^T = \theta B\}.$$

If we make an appropriate choice of g and P then we can diagonalize A in such a way as to preserve one integral entry in $P\nu P^{-1}$.

Lemma 5.5. *Suppose we can choose $\lambda, \mu \in \mathcal{O}$ such that $N(\lambda) = p - 1$, $N(\mu) = p$ and $\text{tr}(r) = 0$ (where $r = \lambda\overline{\mu}$). Then*

$$g_{\lambda,\mu} := \begin{pmatrix} 1 & \lambda \\ 0 & \mu \end{pmatrix} \quad \text{and} \quad P_{\lambda,\mu} = \begin{pmatrix} 1 & \frac{\overline{r}}{p} \\ 0 & 1 \end{pmatrix}$$

are valid choices for g and P .

$$\text{Further } P_{\lambda,\mu}^{-1} = \overline{P_{\lambda,\mu}}.$$

Proof. A simple calculation shows that:

$$A_{\lambda,\mu} = \begin{pmatrix} 1 & \lambda \\ 0 & \mu \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \overline{\lambda} & \overline{\mu} \end{pmatrix} = \begin{pmatrix} 1 + N(\lambda) & \lambda\overline{\mu} \\ \mu\overline{\lambda} & N(\mu) \end{pmatrix} = \begin{pmatrix} p & r \\ \overline{r} & p \end{pmatrix},$$

and also that $\det(A_{\lambda,\mu}) = p^2 - N(r) = p^2 - p(p - 1) = p$ as required.

To prove the second claim we note that $r^2 = -p(p - 1)$ by the Cayley-Hamilton theorem (since $\text{tr}(r) = 0$ and $N(r) = p(p - 1)$). Then

$$P_{\lambda,\mu} A \overline{P_{\lambda,\mu}}^T = \begin{pmatrix} 1 & \frac{\overline{r}}{p} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p & r \\ \overline{r} & p \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{r}{p} & 1 \end{pmatrix} = \begin{pmatrix} p + \frac{r^2}{p} + \frac{\overline{r}}{p}(\text{tr}(r)) & \text{tr}(r) \\ \text{tr}(r) & p \end{pmatrix}$$

and so $P_{\lambda,\mu} A \overline{P_{\lambda,\mu}}^T = \text{diag}(1, p)$.

The final claim follows from the fact that $P_{\lambda,\mu} \overline{P_{\lambda,\mu}} = I$ (which again uses the fact that $\text{tr}(r) = 0$). \square

It is in fact always possible to find **some** maximal order \mathcal{O} of D where such λ, μ exist. For proof of this I refer to an online discussion with John Voight [29], of which the author is grateful. We fix such a choice from now on.

Corollary 5.6. *Let $\nu \in M_2(\mathcal{O})$. Then the bottom left entries of ν and $P_{\lambda,\mu} \nu \overline{P_{\lambda,\mu}}$ are equal (in particular this entry remains in \mathcal{O}).*

Proof. Let $\nu = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ with $\alpha, \beta, \gamma, \delta \in \mathcal{O}$. Then a simple calculation shows that

$$P_{\lambda,\mu} \nu \overline{P_{\lambda,\mu}} = \begin{pmatrix} \alpha + \frac{\overline{r}\gamma}{p} & (\frac{\alpha r}{p} + \beta) + \frac{\overline{r}}{p}(\frac{\gamma r}{p} + \delta) \\ \gamma & \frac{\gamma r}{p} + \delta \end{pmatrix}.$$

\square

The matrix $\eta = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in M_2(D)$ belongs to Z_θ if and only if

$$\eta \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \overline{\eta}^T = \theta \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$$

Equivalently

$$\begin{aligned} N(x) + pN(y) &= \theta \\ N(z) + pN(w) &= \theta p \\ x\bar{z} + py\bar{w} &= 0. \end{aligned}$$

Clearly these equations can have no solutions for $\theta < 0$ and so we only consider $\theta \geq 0$.

A quick calculation shows that $N(x) = N(w)$ and $N(z) = p^2 N(y)$ (a fact we will use soon).

Corollary 5.7. *Let $\theta \geq 0$. Then W_θ consists of all matrices $\nu = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_2(\mathcal{O})^\times$ such that:*

$$\begin{aligned} pN(p\alpha + \bar{r}\gamma) + N(p(\alpha r + p\beta) + \bar{r}(\gamma r + p\delta)) &= \theta p^3 \\ pN(\gamma) + N(\gamma r + p\delta) &= \theta p^2 \\ p\alpha\bar{\gamma} + (\alpha r + p\beta)(\overline{\gamma r + p\delta}) &= -\theta p\bar{r}. \end{aligned}$$

The following algorithm allows us to compute W_θ for $\theta \in \mathbb{N}$. Denote by X_i the subset of \mathcal{O} consisting of norm i elements.

Algorithm 1

Step 0: Set $j := 0$. For each integer $0 \leq i \leq \theta p$, generate the norm lists $X_i, X_{p(\theta p - i)}, X_{p^2 i}$.

Step 1: For each pair of elements $(\gamma, \gamma') \in X_j \times X_{p(\theta p - j)}$ check whether the element $\delta := \frac{\gamma' - \gamma r}{p} \in \mathcal{O}$.

Step 2: For each putative $\gamma \in X_j$ from Step 1 find all elements $\gamma'' \in X_{p(\theta p - j)}$ such that the element $\alpha := \frac{\gamma'' - \bar{r}\gamma}{p} \in \mathcal{O}$.

Step 3: For each putative triple (α, γ, δ) from Step 2 and each $\gamma''' \in X_{p^2 j}$ test whether the element $\beta := \frac{\gamma''' - (\bar{r}(\gamma r + p\delta) + p\alpha r)}{p^2} \in \mathcal{O}$.

Step 4: Check that the entries of each putative tuple from Step 3 satisfies the third equation of Corollary 5.7.

Step 5: Set $j := j + 1$ and repeat steps 1-4 until $j > \theta p$.

Of course once the elements of W_θ have been found it is straight forward to generate the elements of Y_θ by inverting the bijection Φ_θ in Proposition 5.3.

It should be noted that if we run this algorithm for $p = 2$ with the following choices

$$\begin{aligned} D &= \left(\frac{-1, -1}{\mathbb{Q}} \right) \\ \mathcal{O} &= \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z} \frac{1+i+j+k}{2} \\ \lambda &= -1 \\ \mu &= i - k \\ \theta &= 1 \end{aligned}$$

then we get exactly the same elements for $Y_1 = \Gamma^{(2)}$ as Ibukiyama does on p.592 of [18].

5.3. Finding h . We can use mass formulae to get information on class numbers h_1 and h_2 for U_1 and U_2 .

Define the mass of open compact $U \subset \mathrm{GU}_2(D_{\mathbb{A}_f})$ as follows:

$$M(U) := \sum_{m=1}^h \frac{1}{|\Gamma_m|},$$

where $\Gamma_m = \mathrm{GU}_2(D) \cap z_m U z_m^{-1}$ for representatives $z_1, z_2, \dots, z_m \in \mathrm{GU}_2(D_{\mathbb{A}_f})$ of $\mathrm{GU}_2(D) \backslash \mathrm{GU}_2(D_{\mathbb{A}_f}) / U$.

Ibukiyama provides the following formulae for $M(U_1)$ and $M(U_2)$ in [18].

Theorem 5.8. *If D is ramified at p and ∞ then:*

$$M(U_1) = \frac{(p-1)(p^2+1)}{5760},$$

$$M(U_2) = \frac{p^2-1}{5760}.$$

This formula is analogous to the Eichler mass formula and is also a special case of the mass formula of Gan, Hanke and Yu [9].

Proposition 5.9. *$h_1 = 1$ if and only if $|\Gamma^{(1)}| = \frac{5760}{(p-1)(p^2+1)}$. Similarly $h_2 = 1$ if and only if $|\Gamma^{(2)}| = \frac{5760}{p^2-1}$.*

Corollary 5.10. *$h_1 = 1$ if and only if $p = 2, 3$. Similarly $h_2 = 1$ if and only if $p = 2, 3, 5, 7, 11$.*

Proof. A quick calculation shows that the only primes to satisfy $\frac{5760}{(p-1)(p^2+1)} \in \mathbb{N}$ are $p = 2, 3$. Recall $|\Gamma^{(1)}| = 2|\mathcal{O}^\times|^2$. For $p = 2, 3$ we have $|\mathcal{O}^\times| = 24, 12$ respectively and one checks that both values satisfy the equation.

The primes satisfying $\frac{5760}{p^2-1} \in \mathbb{N}$ are $p = 2, 3, 5, 7, 11, 17, 19, 31$. Using Algorithm 1 one finds that $|\Gamma^{(2)}| = \frac{5760}{p^2-1}$ for the cases $p = 2, 3, 5, 7, 11$. \square

Ibukiyama and Hashimoto have produced formulae in [16] and [17] that give the values of h_1 and h_2 for any ramified prime. Their formulae agree with this result.

5.4. Finding the Hecke representatives. Now that we have found an algorithm to generate the elements of $\Gamma^{(2)}$ we consider the same question for the Hecke representatives for the $T_{u,q}$ operator on $A_{j,k-3}(D)$ (where $q \neq p$ is a fixed prime).

Proposition 5.11. *Let D be a quaternion algebra over \mathbb{Q} ramified at p, ∞ for some $p \in \{2, 3, 5, 7, 11\}$. Suppose $u \in \mathrm{GU}_2(D_{\mathbb{A}_f})$ is chosen as in Definition 4.9. Then*

$$U_2 u U_2 = \coprod_{[x_i] \in Y_q / \Gamma^{(2)}} x_i U_2.$$

Proof. Consider an arbitrary decomposition:

$$U_2 u U_2 = \coprod x_i U_2.$$

By Proposition 3.5 we may take $x_i \in \mathrm{GU}_2(D)$ for each i . For the rest of the proof we embed $\mathrm{GU}_2(D) \hookrightarrow \mathrm{GU}_2(D_{\mathbb{A}_f})$ diagonally.

Note that for any prime $l \neq q$ we have

$$U_{2,l} u_l U_{2,l} = U_{2,l} = \mathrm{Stab}_{\mathrm{GU}_2(D_l)}(\mathcal{O}_l^2 g_l) = \mathrm{GU}_2(D_l) \cap g_l^{-1} \mathrm{GL}_2(\mathcal{O}_l) g_l$$

Thus $x_i \in \mathrm{GU}_2(D_l) \cap g_l^{-1} M_2(\mathcal{O}_l)^\times g_l$ and $\mu(x_i) \in \mathbb{Z}_l^\times$ for all i .

To study the behaviour locally at q we fix a choice of $h_q \in \mathrm{GU}_2(D_q)$ such that $\mathcal{O}_q^2 g_q = \mathcal{O}_q^2 h_q$ (which is possible since $\mathcal{O}_q^2 g_q$ is locally equivalent to \mathcal{O}_q^2). Note that $h_q g_q^{-1} \in \mathrm{GL}_2(\mathcal{O}_q)$ so that $h_q = k_q g_q$ for some $k_q \in \mathrm{GL}_2(\mathcal{O}_q) \subseteq M_2(\mathcal{O}_q)^\times$.

Conjugation by h_q gives a bijection between $U_{2,q} u_q U_{2,q}$ and $G(h_q u_q h_q^{-1})G$, where $G = \mathrm{GU}_2(D_q) \cap \mathrm{GL}_2(\mathcal{O}_q)$. If we fix an isomorphism as in Proposition 4.4 then the double coset $G(h_q u_q h_q^{-1})G$ is in bijection with $\mathrm{GSp}_4(\mathbb{Z}_q) M_q \mathrm{GSp}_4(\mathbb{Z}_q)$ (where $M_q = \mathrm{diag}(1, 1, q, q)$).

Since by definition $h_q u_q h_q^{-1} \mapsto M_q \in \mathrm{GSp}_4(\mathbb{Q}_q) \cap M_4(\mathbb{Z}_q)$ we see that $h_q u_q h_q^{-1} \in M_2(\mathcal{O}_q)^\times$ and so $u_q \in \mathrm{GU}_2(D_q) \cap h_q^{-1} M_2(\mathcal{O}_q)^\times h_q$.

However:

$$h_q^{-1} M_2(\mathcal{O}_q)^\times h_q = g_q^{-1} (k_q^{-1} M_2(\mathcal{O}_q)^\times k_q) g_q = g_q^{-1} M_2(\mathcal{O}_q)^\times g_q,$$

thus $u_q \in \mathrm{GU}_2(D_q) \cap g_q^{-1} M_2(\mathcal{O}_q)^\times g_q$ and the same can be said about the x_i .

Also since both the conjugation and our chosen isomorphism respect similitude we find that $\mu(u_q) = \mu(M_q) = q$ and so $\mu(U_{2,q} u_q U_{2,q}) \subseteq q \mathbb{Z}_q^\times$. In particular $\mu(x_i) \in q \mathbb{Z}_q^\times$.

Globally we now see that

$$x_i \in \mathrm{GU}_2(D) \cap \prod_l (\mathrm{GU}_2(D_l) \cap g_l^{-1} M_2(\mathcal{O}_l)^\times g_l) = \mathrm{GU}_2(D) \cap g^{-1} M_2(\mathcal{O})^\times g$$

for each i . We also observe that $\mu(x_i) \in \mathbb{Z} \cap \left(q \mathbb{Z}_q^\times \prod_{l \neq q} \mathbb{Z}_l^\times \right) = \{\pm q\}$. However in our case the similitude is positive definite so that $\mu(x_i) = q$.

Thus the x_i can be taken to lie in Y_q . It is clear that each such element lies in the double coset.

It remains to see which elements of Y_q generate the same left coset. We have $x_i U_2 = x_j U_2$ if and only if $x_j^{-1} x_i \in U_2$. But also $x_i, x_j \in \mathrm{GU}_2(D)$, hence $x_j^{-1} x_i \in \mathrm{GU}_2(D) \cap U_2 = \Gamma^{(2)}$. So equivalence of left cosets is upto right multiplication by $\Gamma^{(2)}$. \square

We have a nice formula for the degree of $T_{u,q}$, found in the work of Ihara [23].

Proposition 5.12. *For $q \neq p$ we have that $\deg(T_{u,q}) = (q+1)(q^2+1)$.*

Employing similar arguments to Proposition 5.11 we get the following:

Proposition 5.13. *Let D be a quaternion algebra over \mathbb{Q} ramified at p, ∞ for some $p \in \{2, 3\}$. Suppose $u \in GU_2(D_{\mathbb{A}_f})$ is chosen as in Definition 4.9. Then*

$$U_1 u U_1 = \coprod_{[x_i] \in (GU_2(D)_q \cap M_2(\mathcal{O})^\times) / \Gamma^{(1)}} x_i U_1.$$

Since $\Gamma^{(1)}$ is given explicitly it is possible to write down explicit representatives in this case.

Corollary 5.14. *Let $n \in \mathbb{N}$. For each $k \in \mathbb{N}$ let $X_k = \{\alpha \in \mathcal{O} \mid N(\alpha) = k\}$, $t_k = |X_k / \mathcal{O}^\times|$ and $x_{1,k}, x_{2,k}, \dots, x_{t_k,k}$ be a set of representatives for X_k / \mathcal{O}^\times . For such a choice of k define:*

$$R_k := \left\{ \begin{pmatrix} x_{i,k} & v \\ w & x_{j,k} \end{pmatrix} \mid \begin{array}{l} 1 \leq i, j \leq t_k, \quad v, w \in X_{n-k} \\ x_{i,k} \bar{w} + v \bar{x}_{j,k} = 0 \end{array} \right\}.$$

The following matrices are representatives for $(GU_2(D)_n \cap M_2(\mathcal{O})^\times) / \Gamma^{(1)}$:

$$\begin{aligned} & \bigcup_{k=m+1}^n R_k, \quad \text{if } n = 2m + 1 \text{ is odd} \\ & \left(\bigcup_{k=m+1}^n R_k \right) \cup R'_m, \quad \text{if } n = 2m \text{ is even.} \end{aligned}$$

The finite subset $R'_m \subset R_m$ is to be constructed in the proof.

Proof. Let $\nu = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_2(\mathcal{O})^\times$. In order for $\nu \in GU_2(D)_n$ to hold we must satisfy the equations:

$$N(\alpha) + N(\beta) = n$$

$$N(\gamma) + N(\delta) = n$$

$$\alpha \bar{\gamma} + \beta \bar{\delta} = 0.$$

In a similar vein to previous discussion these equations imply that $N(\alpha) = N(\delta)$ and $N(\beta) = N(\gamma)$. Note that the first equation implies that $0 \leq N(\alpha) \leq n$.

We wish to study equivalence of these matrices under right multiplication by $\Gamma^{(1)}$.

Case 1: $N(\alpha) \neq N(\beta)$.

We may assume that $N(\alpha) > \frac{n}{2}$ since for $x, y \in \mathcal{O}^\times$:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} = \begin{pmatrix} \beta y & \alpha x \\ \delta y & \gamma x \end{pmatrix}$$

and $N(\beta y) = N(\beta) = n - N(\alpha) > n - \frac{n}{2} = \frac{n}{2}$.

Under this assumption there are no anti-diagonal equivalences so it remains to check for diagonal equivalences.

Now:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} \alpha x & \beta y \\ \gamma x & \delta y \end{pmatrix}.$$

Letting $k = N(\alpha)$ choose $x, y \in \mathcal{O}^\times$ so that $\alpha x = x_{i,k}$ and $\delta y = x_{j,k}$ for some $1 \leq i, j \leq t_k$. Then ν is equivalent to $\begin{pmatrix} x_{i,k} & v \\ w & x_{j,k} \end{pmatrix}$. Clearly the matrices of this form are inequivalent.

It is now clear that R_k gives representatives for the particular subcase $N(\alpha) = k > \frac{n}{2}$.

Case 2: $N(\alpha) = N(\beta) = \frac{n}{2} = m$.

The matrices $\begin{pmatrix} x_{i,m} & v \\ w & x_{j,m} \end{pmatrix}$ may now have extra anti-diagonal equivalences.

Suppose

$$\begin{pmatrix} x_{i,m} & v \\ w & x_{j,m} \end{pmatrix} \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} = \begin{pmatrix} x_{s,m} & v' \\ w' & x_{t,m} \end{pmatrix}.$$

Then x, y are uniquely determined:

$$x = \frac{\bar{w}x_{t,m}}{m}$$

$$y = \frac{\bar{v}x_{s,m}}{m}.$$

Thus each such matrix $\begin{pmatrix} x_{i,m} & v \\ w & x_{j,m} \end{pmatrix}$ with $v, w \in X_m$ can only be equivalent to at most one other matrix:

$$\begin{pmatrix} x_{s,m} & \frac{x_{i,m}\bar{w}x_{t,m}}{m} \\ \frac{x_{j,m}\bar{v}x_{s,m}}{m} & x_{t,m} \end{pmatrix},$$

where $v \sim x_{s,m}$ and $w \sim x_{t,m}$ under the action of right unit multiplication.

Let R'_m be a set consisting of a choice of matrix from each of these equivalence pairs (as $x_{i,m}$ and $x_{j,m}$ run through representatives for X_m/\mathcal{O}^\times and v, w run through elements of X_m satisfying $x_{i,m}\bar{w} + v\bar{x}_{j,m} = 0$). Then it is now clear that R'_m is a set of representatives for this subcase. \square

In the subcase $k = n - 1$ it is often easier to use anti-diagonal equivalence (since $X_1/\mathcal{O}^\times = \{1\}$). In this case we can identify:

$$R_{n-1} \longleftrightarrow \left\{ \begin{pmatrix} 1 & z \\ -\bar{z} & 1 \end{pmatrix} \middle| z \in X_{n-1} \right\}.$$

When $n = 2$ exactly half of these will form a set of representatives. In fact it is simple to see that the equivalent pairs would be:

$$\begin{pmatrix} 1 & z \\ -\bar{z} & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -z \\ \bar{z} & 1 \end{pmatrix}$$

Thus:

$$R'_1 = \left\{ \begin{pmatrix} 1 & z_i \\ -\bar{z}_i & 1 \end{pmatrix} \middle| [z_i] \in \mathcal{O}^\times / \{\pm 1\} \right\}.$$

Example 5.15. If we apply Corollary 5.14 to the choices:

$$D = \left(\frac{-1, -1}{\mathbb{Q}} \right)$$

$$\mathcal{O} = \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z} \frac{1+i+j+k}{2}$$

$$n = 3$$

$$X_3/\mathcal{O}^\times = \{[1 \pm i \pm j]\}$$

we find that Hecke representatives for U_1 with ramified prime $p = 2$ and $q = 3$ are given by:

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \quad x, y \in \{1 \pm i \pm j\}$$

$$\begin{pmatrix} 1 & z \\ -\bar{z} & 1 \end{pmatrix}, \quad z \in \mathcal{O}, N(z) = 2.$$

There are 40 representatives here as expected and they agree with the explicit representatives given by Ibukiyama on p.594 of [18]. \square

So far we have not needed the open compact subgroup U_1 but it is actually of use to us in studying U_2 .

Lemma 5.16. *Let $u \in GU_2(D_{\mathbb{A}_f})$ be chosen to form the $T_{u,q}$ operator with respect to both U_1 and U_2 (for prime $q \neq p$). Then the Hecke representatives for $T_{u,q}$ with respect to U_1 and U_2 can be taken to be the same.*

Proof. Recall that u has identity component away from q and $u_q \notin U_{2,q}$ so the there is only one local condition to check, that $U_{2,q} = U_{1,q}$.

Now $U_2 = \text{Stab}_{GU_2(D_{\mathbb{A}_f})}(\mathcal{O}^2 g)$ where $g \in GL_2(D)$ is chosen so that $\mathcal{O}^2 g$ is in the non-principal genus.

We know that $U_{2,q} = \text{Stab}_{GU_2(D_q)}(\mathcal{O}_q^2 g_q)$. However by construction we know that $\mathcal{O}_q^2 g_q$ is equivalent to \mathcal{O}_q^2 (since $q \neq p$). Thus there exists $h_q \in GU_2(D_q)$ such that $\mathcal{O}_q^2 g_q = \mathcal{O}_q^2 h_q$.

It is then clear that:

$$\text{Stab}_{GU_2(D_q)}(\mathcal{O}_q^2 g_q) = \text{Stab}_{GU_2(D_q)}(\mathcal{O}_q^2 h_q) = \text{Stab}_{GU_2(D_q)}(\mathcal{O}_q^2).$$

Thus $U_{2,q} = U_{1,q}$ and so we are done. \square

This result is useful since we have seen that it is generally easier to generate Hecke representatives for $T_{u,q}$ with respect to U_1 .

Corollary 5.17. *Let the ramified prime of D be $p \in \{2, 3\}$. Then we may use the representatives from Corollary 5.14 as Hecke representatives for $T_{u,q}$ with respect to U_2 (for $q \neq p$).*

Proof. Since $p \in \{2, 3\}$ we know that both the class numbers of U_1, U_2 are 1. Hence both admit rational Hecke representatives.

We also know that given Hecke representatives for $T_{u,q}$ with respect to U_1 we may use them for U_2 . Thus the rational representatives from Corollary 5.14 can be used for U_2 . \square

5.5. Implementing the trace formula. Now that we have algorithms that generate the data needed to use the trace formula we discuss some of the finer details in its implementation, namely how to find character values. Denote by $\chi_{j,k-3}$ the character of the representation $V_{j,k-3}$.

Given $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GU}_2(D)$ we may produce a matrix $A \in \mathrm{GSp}_4(\mathbb{C})$ via the embedding:

$$g \mapsto \begin{pmatrix} \alpha_1 + \alpha_2\sqrt{a} & \beta_1 + \beta_2\sqrt{a} & \alpha_3 + \alpha_4\sqrt{a} & \beta_3 + \beta_4\sqrt{a} \\ \gamma_1 + \gamma_2\sqrt{a} & \delta_1 + \delta_2\sqrt{a} & \gamma_3 + \gamma_4\sqrt{a} & \delta_3 + \delta_4\sqrt{a} \\ b(\alpha_3 - \alpha_4\sqrt{a}) & b(\beta_3 - \beta_4\sqrt{a}) & \alpha_1 - \alpha_2\sqrt{a} & \beta_1 - \beta_2\sqrt{a} \\ b(\gamma_3 - \gamma_4\sqrt{a}) & b(\delta_3 - \delta_4\sqrt{a}) & \gamma_1 - \gamma_2\sqrt{a} & \delta_1 - \delta_2\sqrt{a} \end{pmatrix},$$

where $a = i^2$ and $b = j^2$ in D .

This embedding is the composition of the standard embedding $D^\times \hookrightarrow M_2(K(\sqrt{a}))$ and the isomorphism $\mathrm{GU}_2(M_2(K(\sqrt{a}))) \cong \mathrm{GSp}_4(K(\sqrt{a})) \subseteq \mathrm{GSp}_4(\mathbb{C})$ given in Theorem 4.2.

We know that the image of $\mathrm{GU}_2(\mathbb{H})_1 \cap \mathrm{GU}_2(D)$ under this embedding is a subgroup of $\mathrm{USp}(4)$, so that the matrix $B = \frac{A}{\sqrt{\mu(A)}} \in \mathrm{USp}(4)$. By writing $A = (\sqrt{\mu(A)}I)B$ it follows that:

$$\chi_{j,k-3}(g) = \chi_{j,k-3}(A) = \mu(A)^{\frac{j+2k-6}{2}} \chi_{j,k-3}(B).$$

In order to find $\chi_{j,k-3}(B)$ we first find the eigenvalues of B . This is equivalent to conjugating into the maximal torus of diagonal matrices. Since $B \in \mathrm{USp}(4)$ these eigenvalues will come in two complex conjugate pairs z, \bar{z}, w, \bar{w} for z, w on the unit circle.

The Weyl character formula gives:

$$\chi_{j,k-3}(B) = \frac{w^{j+1}(w^{2(k-2)} - 1)(z^{2(j+k-1)} - 1) - z^{j+1}(z^{2(k-2)} - 1)(w^{2(j+k-1)} - 1)}{(z^2 - 1)(w^2 - 1)(zw - 1)(z - w)(zw)^{j+k-3}}.$$

For any of the cases $z^2 = 1, w^2 = 1, zw = 1, z = w$ one must formally expand this concise formula into a polynomial expression (not an infinite sum since each factor on the denominator except zw divides the numerator). It is easy for a computer package to compute this expansion for a given j, k .

5.6. Finding the trace contribution for the new subspace. Let $\mathrm{tr}(T_{u,q})^{\mathrm{new}}$ and $\mathrm{tr}(T_{u,q})^{\mathrm{old}}$ be the traces of the action of $T_{u,q}$ on $A_{j,k-3}^{\mathrm{new}}(D)$ and $A_{j,k-3}^{\mathrm{old}}(D)$ respectively. Then $\mathrm{tr}(T_{u,q})^{\mathrm{new}} = \mathrm{tr}(T_{u,q}) - \mathrm{tr}(T_{u,q})^{\mathrm{old}}$.

Recall that each eigenform in $A_{j,k-3}^{\mathrm{old}}(D)$ is given by a special pair of eigenforms $F_1 \in \mathcal{A}(D^\times, U, V_j)$ and $F_2 \in \mathcal{A}(\mathrm{GU}_2(D), U_2, V_{j,k-3})$. If $j > 0$ then F_1 corresponds to a unique eigenform in $S_{j+2}^{\mathrm{new}}(\Gamma_0(p))$ by Eichler's correspondence. Attached to the pair (F_1, F_2) is an eigenform $\theta_{F_1 \otimes F_2} \neq 0$ in $M_{j+2k-2}(\mathrm{SL}_2(\mathbb{Z}))$ (it is a cusp form if $j + 2k - 6 \neq 0$).

Let $\alpha_n, \beta_n, \gamma_n$ be the Hecke eigenvalues of $F_1, F_2, \theta_{F_1 \otimes F_2}$ respectively. Ibukiyama links the eigensystems as follows.

Theorem 5.18. *For $q \neq p$ we have the following identity in $\mathbb{C}(t)$:*

$$\sum_{k=0}^{\infty} \beta_q t^k = \frac{1 - q^{j+2k-4}t^2}{(1 - \alpha_q q^{k-2}t + q^{j+2k-3}t^2)(1 - \gamma_q t + q^{j+2k-3}t^2)}.$$

Corollary 5.19. *For $q \neq p$ we have $\beta_q = \gamma_q + q^{k-2}\alpha_q$.*

Ibukiyama conjectures that there is a bijection between pairs of eigenforms (F_1, θ_F) and eigenforms F_2 . With this in mind it is now possible to calculate the oldform trace contribution.

Corollary 5.20. *Suppose $j + 2k - 6 \neq 0$. Let $g_1, g_2, \dots, g_m \in S_{j+2k-2}(SL_2(\mathbb{Z}))$ and $h_1, h_2, \dots, h_n \in S_{j+2}^{new}(\Gamma_0(p))$ be bases of normalized eigenforms with Hecke eigenvalues a_{q,g_i} and a_{q,h_i} respectively.*

Then for $q \neq p$ and $j > 0$:

$$\text{tr}(T_{u,q})^{old} = n \left(\sum_{i=1}^m a_{q,g_i} \right) + mq^{k-2} \left(\sum_{i=1}^n a_{q,h_i} \right).$$

6. EXAMPLES AND SUMMARY

The following table highlights the choices for $D, \mathcal{O}, \lambda, \mu$ that were used.

| p | D | \mathcal{O} | λ | μ |
|-----|---|---|-----------------------------------|---------|
| 2 | $\left(\frac{-1, -1}{\mathbb{Q}} \right)$ | $\mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z} \left(\frac{1+i+j+k}{2} \right)$ | 1 | $i - k$ |
| 3 | $\left(\frac{-1, -3}{\mathbb{Q}} \right)$ | $\mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z} \left(\frac{1+j}{2} \right) \oplus \mathbb{Z} \left(\frac{i+k}{2} \right)$ | $1 + i$ | j |
| 5 | $\left(\frac{-2, -5}{\mathbb{Q}} \right)$ | $\mathbb{Z} \oplus \mathbb{Z} \left(\frac{2-i+k}{4} \right) \oplus \mathbb{Z} \left(\frac{2+3i+k}{4} \right) \oplus \mathbb{Z} \left(\frac{-1+i+j}{2} \right)$ | 2 | j |
| 7 | $\left(\frac{-1, -7}{\mathbb{Q}} \right)$ | $\mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z} \left(\frac{1+j}{2} \right) \oplus \mathbb{Z} \left(\frac{i+k}{2} \right)$ | $2 + \frac{1}{2}i - \frac{1}{2}k$ | j |
| 11 | $\left(\frac{-1, -11}{\mathbb{Q}} \right)$ | $\mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z} \left(\frac{1+j}{2} \right) \oplus \mathbb{Z} \left(\frac{i+k}{2} \right)$ | $1 + 3i$ | j |

Using these choices along with the algorithms and results mentioned previously one can calculate the groups $\Gamma^{(1)}, \Gamma^{(2)}$ for each such p , hence generating tables of dimensions of the spaces $A_{j,k-3}^{new}(D)$. These tables are given in Appendix A.1.

From these tables one isolates 1-dimensional spaces. For each possibility the MAGMA command LRatio allows us to test for large primes dividing Λ_{alg} on the elliptic side. The cases that remained were ones where we expect to find examples of Harder's congruence.

Tables of the congruences observed can be found in Appendix A.2. In particular for $p = 2$ one observes congruences provided in Bergström [1]. We finish with some new examples for $p = 3$.

Example 6.1. By Appendix A.1 we see that

$$\dim(A_{2,5}(D)) = \dim(A_{2,5}^{new}(D)) = \dim(S_{2,8}^{new}(K(3))) = 1.$$

Then $j = 2$ and $k = 8$ so that $j + 2k - 2 = 16$. Let $F \in S_{2,8}^{new}(K(3))$ be the unique normalized eigenform.

One easily checks that $\dim(S_{16}^{new}(\Gamma_0(3))) = 2$. This space is spanned by the two normalized eigenforms with q -expansions:

$$\begin{aligned} f_1(\tau) &= q - 234q^2 - 2187q^3 + 21988q^4 + 280710q^5 + \dots \\ f_2(\tau) &= q - 72q^2 + 2187q^3 - 27584q^4 - 221490q^5 + \dots \end{aligned}$$

Indeed MAGMA informs us that $\text{ord}_{109}(\Lambda_{\text{alg}}(f_1, 10)) = 1$ and so we expect a congruence of the form:

$$b_q \equiv a_q + q^9 + q^6 \pmod{109}$$

for all $q \neq 3$, where b_q are the Hecke eigenvalues of F and a_q the Hecke eigenvalues of f_1 . As discussed earlier we will only work with the case $q = 2$ for simplicity.

The algorithms mentioned earlier then calculate the necessary $\frac{5760}{3^2-1} = 720$ matrices belonging to $\Gamma^{(2)}$ and the $(2+1)(2^2+1) = 15$ Hecke representatives for the operator $T_{u,2}$. Applying the trace formula we find that $\text{tr}(T_{u,2}) = -312$.

Now since $A_{2,5}(D) = A_{2,5}^{\text{new}}(D)$ we have that $\text{tr}(T_{u,2}) = \text{tr}(T_{u,2})^{\text{new}}$. Also the spaces are 1-dimensional and so in fact $b_2 = \text{tr}(T_{u,2})^{\text{new}} = -312$.

The congruence is then simple to check:

$$-312 \equiv -234 + 2^9 + 2^6 \pmod{109}.$$

□

Example 6.2. We see an example where we must subtract off the oldform contribution from the trace. By Appendix A.1 we see that

$$\dim(A_{8,2}(D)) = 3$$

whereas

$$\dim(A_{8,2}^{\text{new}}(D)) = \dim(S_{8,5}^{\text{new}}(K(3))) = 1.$$

Then $j = 8$ and $k = 5$ so that $j + 2k - 2 = 16$ again. Let $F \in S_{8,5}^{\text{new}}(K(3))$ be the unique normalized eigenform.

MAGMA informs us that $\text{ord}_{67}(\Lambda_{\text{alg}}(f_2, 13)) = 1$ and so we expect a congruence of the form:

$$b_q \equiv a_q + q^{12} + q^3 \pmod{67}$$

for all $q \neq 3$.

Applying the trace formula this time gives $\text{tr}(T_{u,2}) = 300$. However since $\dim(A_{8,2}(D)) > \dim(A_{8,2}^{\text{new}}(D))$ there is an oldform contribution to this trace. In order to find it we need Hecke eigenvalues of normalized eigenforms for the spaces $S_{16}(SL_2(\mathbb{Z}))$ and $S_{10}^{\text{new}}(\Gamma_0(3))$.

It is known that $\dim(S_{16}(SL_2(\mathbb{Z}))) = 1$ and that the unique normalized eigenform has q -expansion:

$$g(\tau) = q + 216q^2 - 3348q^3 + 13888q^4 + 52110 + \dots$$

Also $\dim(S_{10}^{\text{new}}(\Gamma_0(3))) = 2$ and the normalized eigenforms have the following q -expansions:

$$\begin{aligned} h_1(\tau) &= q - 36q^2 - 81q^3 + 784q^4 - 1314q^5 + \dots \\ h_2(\tau) &= q + 18q^2 + 81q^3 - 188q^4 - 1540q^5 + \dots \end{aligned}$$

Thus using Corollary 5.20 the oldform contribution is:

$$\begin{aligned} \text{tr}(T_{u,2})^{\text{old}} &= 2a_{2,g} + 2^3(a_{2,h_1} + a_{2,h_2}) = 512 + 8(-36 + 18) \\ &= 288 \end{aligned}$$

Hence $\text{tr}(T_{u,2})^{\text{new}} = \text{tr}(T_{u,2}) - \text{tr}(T_{u,2})^{\text{old}} = 300 - 288 = 12$. Since our space of algebraic forms is 1-dimensional we must have $b_2 = \text{tr}(T_{u,2})^{\text{new}} = 12$.

The congruence is then simple to check:

$$12 \equiv -72 + 2^{12} + 2^3 \pmod{67}.$$

□

Example 6.3. Our final example is a case where the Hecke eigenvalues of the elliptic modular form lie in a quadratic extension of \mathbb{Q} .

By Appendix A.1 we see that

$$\dim(A_{6,2}(D)) = \dim(A_{6,2}^{\text{new}}(D)) = \dim(S_{6,5}^{\text{new}}(K(3))) = 1.$$

Then $j = 6$ and $k = 5$ so that $j + 2k - 2 = 14$. Let $F \in S_{6,5}^{\text{new}}(K(3))$ be the unique normalized eigenform.

One easily checks that $\dim(S_{14}^{\text{new}}(\Gamma_0(3))) = 3$. This space is spanned by the three normalized newforms with q -expansions:

$$\begin{aligned} f_1(\tau) &= q - 12q^2 - 729q^3 + \dots \\ f_2(\tau) &= q - (27 + 3\sqrt{1969})q^2 + 729q^3 + \dots \\ f_3(\tau) &= q - (27 - 3\sqrt{1969})q^2 + 729q^3 + \dots \end{aligned}$$

MAGMA informs us that $\text{ord}_{47}(N_{\mathbb{Q}(\sqrt{1969})/\mathbb{Q}}(\Lambda_{\text{alg}}(f_2, 11))) = 1$ and so we expect a congruence of the form:

$$b_q \equiv a_q + q^{10} + q^3 \pmod{\lambda}$$

for some prime ideal λ of $\mathbb{Z}\left[\frac{1+\sqrt{1969}}{2}\right]$ satisfying $\lambda \mid 47$ (note that 47 splits in this extension).

The trace formula gives $\text{tr}(T_{u,2}) = 72$ and the usual arguments show that $b_2 = 72$. It is then observed that

$$N_{\mathbb{Q}(\sqrt{1969})/\mathbb{Q}}(b_2 - a_2 - 2^{10} - 2^3) = N_{\mathbb{Q}(\sqrt{1969})/\mathbb{Q}}(-933 + 3\sqrt{1969}) = 852768$$

This is divisible by 47 and so the congruence holds for $q = 2$. \square

APPENDIX A. TABLES

A.1. Newform dimensions. For each prime $p = 2, 3, 5, 7, 11$ the following tables give values of $\dim(A_{j,k}^{\text{new}}(D))$ for $0 \leq j \leq 20$ even and $0 \leq k \leq 15$. We use the specific quaternion algebras given in section 6. Note that Ibukiyama conjectures that these values are equal to $\dim(S_{j,k+3}^{\text{new}}(K(p)))$.

| | | | | | | | | | | | | | | | | |
|----------|----|-----|-----|-----|-----|-----|-----|------|------|------|------|------|------|------|------|------|
| $p = 2$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| 0 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 2 | 1 | 2 | 1 | 3 | 2 | 3 | 2 |
| 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 4 | 2 | 4 | 5 |
| 4 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 3 | 3 | 4 | 4 | 7 | 7 | 9 | 9 |
| 6 | 0 | 0 | 0 | 0 | 1 | 0 | 2 | 2 | 4 | 3 | 5 | 7 | 10 | 9 | 13 | 14 |
| 8 | 0 | 0 | 0 | 1 | 2 | 2 | 2 | 4 | 7 | 7 | 9 | 10 | 15 | 17 | 20 | 22 |
| 10 | 0 | 0 | 0 | 1 | 3 | 4 | 4 | 6 | 10 | 10 | 14 | 17 | 21 | 23 | 29 | 33 |
| 12 | 0 | 0 | 1 | 1 | 3 | 5 | 6 | 8 | 12 | 14 | 17 | 21 | 28 | 30 | 37 | 41 |
| 14 | 0 | 0 | 1 | 3 | 5 | 6 | 9 | 12 | 17 | 19 | 24 | 29 | 37 | 40 | 49 | 56 |
| 16 | 0 | 1 | 2 | 4 | 8 | 9 | 13 | 16 | 23 | 26 | 32 | 38 | 48 | 53 | 63 | 70 |
| 18 | 0 | 0 | 2 | 5 | 9 | 11 | 15 | 20 | 28 | 31 | 39 | 46 | 58 | 64 | 76 | 86 |
| 20 | 0 | 2 | 3 | 7 | 12 | 16 | 20 | 26 | 35 | 41 | 50 | 58 | 71 | 81 | 94 | 106 |
| $p = 3$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| 0 | 1 | 0 | 0 | 1 | 1 | 1 | 2 | 1 | 2 | 3 | 3 | 3 | 5 | 4 | 5 | 8 |
| 2 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 2 | 2 | 4 | 4 | 6 | 8 | 9 | 11 | 14 |
| 4 | 0 | 0 | 0 | 1 | 0 | 2 | 3 | 3 | 5 | 8 | 8 | 12 | 15 | 17 | 22 | 27 |
| 6 | 0 | 0 | 1 | 2 | 2 | 3 | 7 | 7 | 10 | 14 | 16 | 21 | 27 | 30 | 37 | 45 |
| 8 | 0 | 0 | 1 | 3 | 4 | 6 | 8 | 12 | 16 | 20 | 25 | 31 | 38 | 46 | 54 | 64 |
| 10 | 0 | 0 | 1 | 4 | 5 | 10 | 13 | 16 | 23 | 30 | 35 | 45 | 54 | 63 | 76 | 90 |
| 12 | 0 | 1 | 4 | 7 | 8 | 15 | 20 | 25 | 32 | 43 | 49 | 62 | 75 | 86 | 102 | 121 |
| 14 | 0 | 1 | 5 | 9 | 13 | 19 | 27 | 34 | 44 | 55 | 67 | 81 | 97 | 113 | 133 | 154 |
| 16 | 0 | 2 | 6 | 13 | 17 | 25 | 36 | 44 | 57 | 72 | 84 | 104 | 124 | 142 | 167 | 194 |
| 18 | 1 | 3 | 10 | 18 | 24 | 35 | 47 | 58 | 75 | 93 | 109 | 131 | 157 | 180 | 209 | 242 |
| 20 | 0 | 6 | 12 | 22 | 31 | 45 | 58 | 74 | 92 | 114 | 136 | 162 | 189 | 221 | 254 | 292 |
| $p = 5$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| 0 | 1 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 5 | 5 | 7 | 8 | 10 | 11 | 14 | 16 |
| 2 | 0 | 0 | 0 | 1 | 2 | 2 | 3 | 5 | 7 | 10 | 13 | 17 | 22 | 27 | 33 | 40 |
| 4 | 0 | 1 | 1 | 3 | 4 | 7 | 10 | 14 | 18 | 25 | 31 | 39 | 48 | 59 | 70 | 84 |
| 6 | 0 | 0 | 3 | 4 | 7 | 11 | 17 | 22 | 31 | 39 | 50 | 63 | 77 | 92 | 112 | 131 |
| 8 | 0 | 3 | 5 | 9 | 15 | 21 | 28 | 40 | 51 | 64 | 81 | 99 | 119 | 144 | 169 | 198 |
| 10 | 0 | 2 | 6 | 12 | 20 | 29 | 41 | 54 | 71 | 90 | 112 | 136 | 165 | 196 | 231 | 270 |
| 12 | 1 | 6 | 14 | 22 | 31 | 48 | 62 | 81 | 104 | 130 | 157 | 193 | 228 | 269 | 316 | 366 |
| 14 | 0 | 7 | 17 | 27 | 44 | 60 | 82 | 107 | 136 | 167 | 207 | 247 | 294 | 346 | 404 | 465 |
| 16 | 3 | 13 | 24 | 43 | 61 | 84 | 113 | 145 | 180 | 224 | 269 | 322 | 381 | 445 | 514 | 594 |
| 18 | 3 | 14 | 34 | 53 | 78 | 109 | 143 | 181 | 230 | 279 | 336 | 401 | 472 | 548 | 636 | 727 |
| 20 | 4 | 26 | 45 | 72 | 105 | 143 | 183 | 236 | 289 | 352 | 423 | 500 | 582 | 680 | 779 | 890 |
| $p = 7$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| 0 | 1 | 1 | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 10 | 13 | 15 | 18 | 22 | 26 | 31 |
| 2 | 0 | 0 | 1 | 1 | 3 | 5 | 8 | 12 | 16 | 22 | 29 | 37 | 47 | 57 | 70 | 84 |
| 4 | 0 | 1 | 1 | 5 | 7 | 12 | 18 | 26 | 34 | 47 | 59 | 75 | 93 | 114 | 136 | 164 |
| 6 | 1 | 3 | 7 | 11 | 18 | 26 | 38 | 50 | 67 | 85 | 107 | 133 | 162 | 194 | 232 | 272 |
| 8 | 0 | 6 | 10 | 19 | 29 | 43 | 57 | 80 | 102 | 130 | 162 | 199 | 239 | 289 | 339 | 398 |
| 10 | 1 | 5 | 14 | 26 | 42 | 60 | 85 | 111 | 145 | 183 | 228 | 276 | 334 | 396 | 467 | 545 |
| 12 | 4 | 15 | 29 | 47 | 67 | 98 | 128 | 168 | 212 | 265 | 321 | 391 | 463 | 546 | 638 | 740 |
| 14 | 4 | 18 | 38 | 60 | 93 | 127 | 171 | 221 | 280 | 344 | 422 | 504 | 599 | 703 | 819 | 943 |
| 16 | 5 | 27 | 49 | 86 | 122 | 170 | 226 | 291 | 361 | 449 | 539 | 646 | 762 | 892 | 1030 | 1189 |
| 18 | 13 | 37 | 76 | 116 | 168 | 228 | 299 | 377 | 473 | 573 | 690 | 818 | 962 | 1116 | 1291 | 1475 |
| 20 | 13 | 54 | 94 | 150 | 214 | 291 | 373 | 477 | 585 | 712 | 852 | 1008 | 1174 | 1367 | 1567 | 1791 |
| $p = 11$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| 0 | 1 | 1 | 2 | 3 | 4 | 6 | 8 | 11 | 15 | 19 | 24 | 31 | 38 | 46 | 56 | 67 |
| 2 | 0 | 1 | 2 | 4 | 9 | 14 | 21 | 31 | 43 | 57 | 75 | 95 | 119 | 147 | 178 | 213 |
| 4 | 1 | 4 | 6 | 15 | 22 | 35 | 51 | 71 | 93 | 125 | 157 | 197 | 243 | 296 | 353 | 422 |
| 6 | 3 | 5 | 18 | 27 | 44 | 66 | 94 | 124 | 168 | 212 | 268 | 332 | 405 | 484 | 581 | 681 |
| 8 | 2 | 17 | 28 | 49 | 77 | 111 | 149 | 205 | 261 | 331 | 413 | 506 | 607 | 730 | 858 | 1005 |
| 10 | 7 | 20 | 43 | 75 | 115 | 161 | 225 | 293 | 377 | 475 | 586 | 709 | 856 | 1012 | 1189 | 1386 |
| 12 | 11 | 38 | 74 | 120 | 170 | 248 | 342 | 422 | 536 | 667 | 808 | 983 | 1163 | 1372 | 1603 | 1857 |
| 14 | 15 | 53 | 103 | 159 | 243 | 329 | 439 | 567 | 714 | 875 | 1072 | 1278 | 1515 | 1778 | 2068 | 2379 |
| 16 | 26 | 78 | 138 | 230 | 324 | 444 | 586 | 749 | 928 | 1147 | 1377 | 1642 | 1937 | 2261 | 2610 | 3008 |
| 18 | 38 | 100 | 198 | 298 | 428 | 582 | 759 | 954 | 1195 | 1447 | 1738 | 2063 | 2421 | 2806 | 3246 | 3707 |
| 20 | 44 | 148 | 252 | 390 | 554 | 745 | 954 | 1215 | 1487 | 1804 | 2157 | 2547 | 2966 | 3448 | 3951 | 4509 |

A.2. Congruences. The following table gives information on the congruences found. For simplicity we only give the Hecke eigenvalues at $q = 3$ when $p = 2$ and $q = 2$ when $p = 3, 5, 7, 11$.

Note that there is no congruence at level 11 (even though one is expected). This does not contradict the conjecture since $\lambda \nmid 11$ in this case.

Whenever a_q is rational we give the Hecke eigenvalue explicitly. When it lies in a bigger number field we give the minimal polynomial $f(x)$ defining \mathbb{Q}_f (then the Hecke eigenvalue a_2 in all of our cases is exactly a root α of this polynomial).

The large primes given are the rational primes lying below the prime for which the congruence holds.

| | (j, k) | $N(\lambda)$ | $\text{tr}(T_q)$ | b_q | a_q |
|----------|----------|--------------|------------------|---------|----------------------------------|
| $p = 2$ | (0, 14) | 37 | 2223720 | 2223720 | 97956 |
| | (2, 10) | 61 | 18360 | 18360 | -13092 |
| | (2, 11) | 71 | -57528 | -57528 | 59316 |
| | (2, 12) | 29 | -122040 | -122040 | -505908 |
| | (4, 10) | 61 | -189720 | -189720 | 71604 |
| | (6, 7) | 29 | 1872 | 3240 | 6084 |
| | (10, 6) | 109 | 216 | 216 | -13092 |
| | (12, 5) | 79 | 77544 | -7560 | -53028 |
| | (12, 6) | 23 | -275688 | 30600 | 71604 |
| | (14, 5) | 379 | 102960 | 63000 | 59316 |
| | (16, 4) | 37 | -97488 | -23400 | 71604 |
| $p = 3$ | (2, 8) | 109 | -312 | -312 | -234 |
| | (4, 6) | 23 | -36 | -36 | -12 |
| | (6, 5) | 47 | 72 | 72 | $x^2 + 54x - 16992$ |
| | (8, 5) | 67 | 300 | 12 | -72 |
| | (10, 5) | 433 | 120 | 24 | $x^2 - 594x - 42912$ |
| | (12, 4) | 23 | -1716 | 132 | 204 |
| | (14, 4) | 617 | -240 | 72 | $x^2 - 702x - 664128$ |
| $p = 5$ | (2, 7) | 61 | -76 | -76 | $x^3 - 142x^2 - 11144x + 901248$ |
| $p = 7$ | (2, 5) | 263 | -44 | -44 | $x^3 - 21x^2 - 1326x + 19080$ |
| | (4, 4) | 101 | -2 | -2 | $x^2 + 6x - 184$ |
| | (4, 5) | 43 | -70 | 10 | $x^2 + 54x - 2640$ |
| $p = 11$ | (2, 4) | 11 | -20 | -20 | N/A |

REFERENCES

- [1] J. Bergström, C. Faber, G. van der Geer, *Siegel Modular Forms of Genus 2 and Level 2: Cohomological Computations and Conjectures*. Int. Math. Res. Not. IMRN, 2008.
- [2] S. Böcherer, R. Schulze-Pillot, *Vector valued theta series and Waldspurger's theorem*. Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg, Volume 64, Issue 1, pp 211 – 233, 1994.
- [3] J. H. Bruinier, G. van der Geer, G. Harder, D. Zagier, *The 1-2-3 of Modular Forms*. Springer, Universitext, 2008.
- [4] K. Buzzard, *Computing modular forms on definite quaternion algebras*. http://www.imperial.ac.uk/~buzzard/maths/research/notes/old_notes_about_computing_modular_forms_on_def_quat_algs.pdf
- [5] G. Chenevier, J. Lannes, *Formes automorphes et voisins de Kneser des réseaux de Niemeier*. <http://arxiv.org/pdf/1409.7616v2.pdf>.
- [6] L. Dembelé, *On the computation of algebraic modular forms on compact inner forms of GSp_4* . Math. Comp. 83, 288, 1931 – 1950, 2014.
- [7] N. Dummigan, D. Fretwell, *Ramanujan style congruences of local origin*. Journal of Number Theory, Volume 143, Pages 248 – 261, 2014.

- [8] N. Dummigan, *A simple trace formula for algebraic modular forms*. Experimental Mathematics, 22, 123 – 131, 2013.
- [9] W. T. Gan, J. P. Hanke, J. Yu, *On an exact mass formula of Shimura*. Duke Math. J. Volume 107, 1, 103 – 133, 2001.
- [10] A. Ghitza, *Hecke eigenvalues of Siegel modular forms (mod p) and of algebraic modular forms*. Journal of Number Theory 106, 2, 345 – 384, 2004.
- [11] A. Ghitza, N. Ryan, D. Sulon, *Computations of vector-valued Siegel modular forms*. Journal of Number Theory, Volume 133, Issue 11, Pages 3921 – 3940, 2013.
- [12] M. Greenberg, J. Voight, *Lattice methods for algebraic modular forms on classical groups*. Computations with Modular Forms, Contributions in Mathematical and Computational Sciences Volume 6, 147 – 179, 2014.
- [13] B. Gross, *Algebraic Modular Forms* Israel Journal of Mathematics, Volume 113, Issue 1, 61 – 93, 1999.
- [14] G. Harder, *A congruence between a Siegel and an Elliptic Modular Form*. Featured in “The 1-2-3 of Modular Forms”.
- [15] G. Harder, *Cohomology in the language of Adeles*.
<http://www.math.uni-bonn.de/people/harder/Manuscripts/buch/chap3-2014.pdf>.
- [16] K. Hashimoto and T. Ibukiyama, *On class numbers of positive definite binary quaternion hermitian forms*. J.Fac.Sci.Univ.Tokyo Sect.IA Math., 27, 549 – 601, 1980.
- [17] K. Hashimoto and T. Ibukiyama, *On class numbers of positive definite binary quaternion hermitian forms (II)*. J.Fac.Sci.Univ.Tokyo Sect.IA Math., 28, 695 – 699, 1982.
- [18] T. Ibukiyama, *On symplectic Euler factors of genus 2*. Proc. Japan Acad. Ser. A Math. Sci. Volume 57, Number 5, 271 – 275, 1981.
- [19] T. Ibukiyama, *On maximal orders of division quaternion algebras over the rational number field with certain optimal embeddings*. Nagoya Math. J. Volume 88, 181 – 195, 1982.
- [20] T. Ibukiyama, *On Automorphic Forms on the Unitary Symplectic Group $Sp(n)$ and $SL_2(\mathbb{R})$* . Mathematische Annalen, Volume 278, Issue 1 – 4, 307 – 327, 1987.
- [21] T. Ibukiyama, T. Katsura, F. Oort, *Supersingular curves of genus 2 and class numbers*. Compositio Mathematica, Volume 57, Issue 2, 127 – 152, 1986.
- [22] T. Ibukiyama, *Paramodular forms and compact twist*. Automorphic Forms on $GSp(4)$, Proceedings of the 9th Autumn Workshop on Number Theory, Ed. M. Furusawa, 37 – 48, 2007.
- [23] Y. Ihara, *On certain arithmetical Dirichlet series*. J. Math. Soc. Japan, 16, 214 – 225, 1964.
- [24] D. H. Lehmer, *The vanishing of Ramanujans function $\tau(n)$* . Duke Math. J. 14: 429433, 1947.
- [25] B. Roberts, R. Schmidt, *Local Newforms for GSp_4* . Springer, Lecture notes in mathematics 1918, 2007.
- [26] G. Shimura, *Arithmetic of alternating forms and quaternion hermitian forms*. J. Math. Soc. Japan, Volume 15, Number 1, 33 – 65, 1963.
- [27] W. Stein, *Modular Forms, a Computational Approach*. American Mathematical Society, GSM 79, 2007.
- [28] G. Wiese *Galois Representations*
<http://math.uni.lu/~wiese/notes/GalRep.pdf>
- [29] <http://mathoverflow.net/questions/159604/integral-elements-of-quaternion-algebras-with-predescribed-properties>.